Approximation of the biharmonic problem using P1 finite elements

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May 24, 2013

Abstract

We study in this paper a P1 finite element approximation of the solution in $H_0^2(\Omega)$ of a biharmonic problem. Since the P1 finite element method only leads to an approximate solution in $H_0^1(\Omega)$, a discrete Laplace operator is used in the numerical scheme. The convergence of the method is shown, for the general case of a solution with $H_0^2(\Omega)$ regularity, thanks to compactness results and to the use of a particular interpolation of regular functions with compact supports. An error estimate is proved in the case where the solution is in $C^4(\overline{\Omega})$. The order of this error estimate is equal to 1 if the solution has a compact support, and only 1/5 otherwise. Numerical results show that these orders are not sharp in particular situations.

1 Introduction

This paper deals with the approximation of the following problem, called the biharmonic problem, which arises in various frameworks of fluid or solid mechanics:

$$\begin{aligned}&\text{find } u \text{ such that } \Delta(\Delta u)(\boldsymbol{x}) = f(\boldsymbol{x}) - \text{div}\boldsymbol{g}(\boldsymbol{x}) + \Delta\ell(\boldsymbol{x}),\ \text{ for } \boldsymbol{x} \in \Omega,\\ &u(\boldsymbol{x}) = 0 \text{ and } \nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_{\partial\Omega}(\boldsymbol{x}) = 0,\ \text{ for } \boldsymbol{x} \in \partial\Omega.\end{aligned} \tag{1}$$

In this paper, Problem (1) is considered in the following weak sense:

$$\begin{aligned}&\text{find } u \in H_0^2(\Omega) \text{ such that}\\ &\forall v \in H_0^2(\Omega),\ \int_\Omega \Delta u(\boldsymbol{x})\Delta v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega (f(\boldsymbol{x})v(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) + \ell(\boldsymbol{x})\Delta v(\boldsymbol{x}))\mathrm{d}\boldsymbol{x},\end{aligned} \tag{2}$$

where $H_0^2(\Omega)$ denotes the closure in $H^2(\Omega)$ of the set $C_c^\infty(\Omega)$ of infinitely continuously differentiable functions with compact support in Ω, and where

$$\begin{aligned}&d \in \mathbb{N} \setminus \{0\} \text{ denotes the space dimension,}\\ &\Omega \text{ is an open polygonal bounded and connected subset of } \mathbb{R}^d,\\ &\text{with Lipschitz-continuous boundary } \partial\Omega,\end{aligned} \tag{3}$$

and

$$f \in L^2(\Omega), \quad \ell \in L^2(\Omega) \text{ and } \boldsymbol{g} \in (L^2(\Omega))^d. \tag{4}$$

Numerous discretization methods for Problem (2) have been proposed in the recent past. The most classical is probably the conforming finite element method; the finite element space must then be a finite dimensional subspace of the Sobolev space $H^2(\Omega)$. Hence elementary basis functions are sought such that the reconstructed global basis functions on Ω belong to $C^1(\overline{\Omega})$. On Cartesian meshes, such basis functions are found by generalizing the one-dimensional P^3 Hermite finite element to the multi-dimensional framework. This task becomes much more difficult on more general meshes and involves rather sophisticated finite elements such as the Argyris finite element on triangles in 2D, which unfortunately requires 21 degrees of freedom [8]. Hence non-conforming FEMs have also been widely studied: see e.g. [8, Section 49],

[9], and references therein, and [3, 4] for more recent works. Discontinuous Galerkin methods have also been recently developed and analysed [12, 13, 14, 11]; error estimates have been derived for polynomials of degree greater or equal to two or three. Other methods which have been developed for fourth order problems include mixed methods [6, 12] (see also references therein), [15], and compact finite difference methods [7, 2, 1]. All the above methods are high order methods, and therefore, rather computationally expensive and may not be so easy to implement. Recently, a cheaper low order method based on the discretization of the Laplace operator by a cell centred finite volume scheme was proposed [10].

The idea developed in the present paper is to use the discretization of the Laplace operator, which is naturally provided by the continuous piecewise linear finite element method (see Section 2). Such a method leads to a nonconforming method in $H^2(\Omega)$, since it only provides an approximate solution in $H^1_0(\Omega)$. This discrete Laplace operator is then used in a discrete bilinear form, which is applied to the elements of the P1 finite element space which vanish at the boundary. Note that the condition $\nabla u \cdot \boldsymbol{n}_{\partial\Omega} = 0$ at the boundary of the domain is satisfied by the limit of a sequence of approximate solutions thanks to the definition of the discrete Laplace operator (see Lemma 3.4 in Section 3). Then the convergence has to be proved, using a suitable interpolation of regular test functions. In the general case, the discrete Laplace operator applied to the standard interpolation of a regular function is not consistent with the continuous Laplace operator applied to this function, although it leads to a second order discrete operator (see Lemma 3.5). Therefore a non standard interpolation of regular functions in the P1 finite element space has to be derived (see Lemma 3.6). Error estimates are derived in the case where the continuous solution has some regularity (Section 4). An order 1 is shown in the case where the solution has a compact support in Ω (Theorem 4.2), but only 1/5 for a general regular solution (Theorem 4.3). It is worth noticing that the order of these estimates is lower than that which is numerically observed in various situations (Section 5).

A short conclusion of ongoing research is finally drawn in Section 6.

2 Definition of the scheme

Let Ω be an open polyhedral domain in $\mathbb{R}^d$, with $d \in \mathbb{N}^\star$. We consider a conforming simplicial mesh $\mathcal{T}$ of Ω (in the standard sense provided for example in [8]). We denote by $h_\mathcal{T}$ the maximum of the diameters h_S of all $S \in \mathcal{T}$.

Let $\mathcal{V}$ be the finite set of the vertices ot the mesh; the set of the vertices of a simplex $S \in \mathcal{T}$ is denoted by $\mathcal{V}_S$, and the subset of all $S \in \mathcal{T}$ such that $\boldsymbol{z} \in \mathcal{V}_S$ is denoted by $\mathcal{T}_{\boldsymbol{z}}$. We denote by $\mathcal{V}_{\rm ext}$ the set of all $\boldsymbol{z} \in \mathcal{V}$ such that $\boldsymbol{z} \in \partial\Omega$, and we denote $\mathcal{V}_{\rm int} = \mathcal{V} \setminus \mathcal{V}_{\rm ext}$ the set of the interior vertices.

Let $(K_{\boldsymbol{z}})_{\boldsymbol{z}\in\mathcal{V}}$ be a family of disjoint open connected subsets of Ω such that:

1. for all $\boldsymbol{z} \in \mathcal{V}$, $\boldsymbol{z} \in K_{\boldsymbol{z}}$ and $K_{\boldsymbol{z}} \subset \bigcup_{S\in\mathcal{T}_{\boldsymbol{z}}} S$,
2. $\bigcup_{\boldsymbol{z}\in\mathcal{V}} \overline{K_{\boldsymbol{z}}} = \overline{\Omega}$.

We denote by $\theta_\mathcal{T}$ the minimum of $(\theta_S)_{S\in\mathcal{T}}, (\theta_{\boldsymbol{z}})_{\boldsymbol{z}\in\mathcal{V}}$, where θ_S is the ratio between the radius of the largest Euclidean ball contained in S and $\operatorname{diam}(S)$, and

$$\theta_{\boldsymbol{z}} = \frac{|K_{\boldsymbol{z}}|}{|\bigcup_{S\in\mathcal{T}_{\boldsymbol{z}}} S|},\ \forall \boldsymbol{z} \in \mathcal{V}.$$

For any $\boldsymbol{z} \in \mathcal{V}$, we denote by $\mathcal{V}_{\boldsymbol{z}}$ the set of all $\boldsymbol{y} \in \mathcal{V}$ such that there exists $S \in \mathcal{T}_{\boldsymbol{z}}$ with $\boldsymbol{y} \in \mathcal{V}_S$ (which means that $\mathcal{V}_{\boldsymbol{z}} = \bigcup_{S\in\mathcal{T}_{\boldsymbol{z}}} \mathcal{V}_S$). In the following, we will also use the notation $\mathcal{T}$ for denoting the whole set of discrete geometric definitions.

Remark 1 The results of this paper hold under the general requirements on $K_{\boldsymbol{z}}$ given above. In the numerical examples given in this paper, we use the following definition of $K_{\boldsymbol{z}}$. For all $S \in \mathcal{T}$ and all $\boldsymbol{z} \in \mathcal{V}_S$, we denote by $K_{S,\boldsymbol{z}}$ the subset of S of all points whose barycentric coordinate related to $\boldsymbol{z}$ is larger than that related to any $\boldsymbol{z}' \in \mathcal{V}_S$ with $\boldsymbol{z}' \neq \boldsymbol{z}$ (see figure 1). We then denote for all $\boldsymbol{z} \in \mathcal{V}$ by $K_{\boldsymbol{z}}$ the union of all $K_{S,\boldsymbol{z}}$, for all $S \in \mathcal{T}_{\boldsymbol{z}}$.

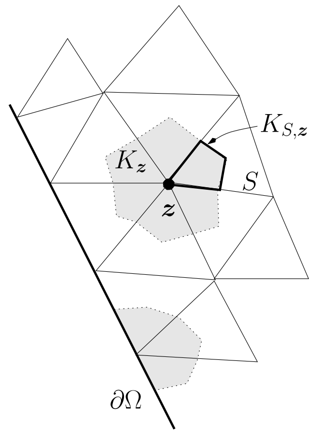


Figure 1: Definition of $K_{\boldsymbol{z}}$

For any $\boldsymbol{z} \in \mathcal{V}$, let $\xi_{\boldsymbol{z}} \in H^1(\Omega)$ be the piecewise affine basis function of the P1 finite element, such that $\xi_{\boldsymbol{z}}(\boldsymbol{z}) = 1$ and $\xi_{\boldsymbol{z}}(\boldsymbol{z}') = 0$ for all $\boldsymbol{z}' \in \mathcal{V} \setminus \{\boldsymbol{z}\}$. We then denote by $V_{\mathcal{T}}$ the vector space spanned by all functions $\xi_{\boldsymbol{z}}$, $\boldsymbol{z} \in \mathcal{V}$. We remark that the property $v = \sum_{\boldsymbol{z}\in\mathcal{V}} v(\boldsymbol{z})\xi_{\boldsymbol{z}}$ holds for any $v \in V_{\mathcal{T}}$, and we define the natural P1 interpolation $\mathcal{I}_{\mathcal{T}}\varphi \in V_{\mathcal{T}}$ of any continuous function φ by

$$\mathcal{I}_{\mathcal{T}}\varphi(\boldsymbol{z}) = \varphi(\boldsymbol{z}),\ \forall \boldsymbol{z} \in \mathcal{V},\ \forall \varphi \in C^0(\overline{\Omega}). \tag{5}$$

For any $u \in V_{\mathcal{T}}$ and any $S \in \mathcal{T}$, we denote by $\nabla_S u$ the constant value of ∇u in S, and we define

$$T_{\boldsymbol{z}\boldsymbol{y}} = -\int_\Omega \nabla\xi_{\boldsymbol{z}}(\boldsymbol{x}) \cdot \nabla\xi_{\boldsymbol{y}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\sum_{S\in\mathcal{T}} |S|\nabla_S\xi_{\boldsymbol{z}} \cdot \nabla_S\xi_{\boldsymbol{y}},\ \forall \boldsymbol{z}, \boldsymbol{y} \in \mathcal{V}. \tag{6}$$

Using the property $\sum_{\boldsymbol{y}\in\mathcal{V}} T_{\boldsymbol{z}\boldsymbol{y}} = 0$ since $\sum_{\boldsymbol{y}\in\mathcal{V}} \xi_{\boldsymbol{y}}(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \Omega$, let $\Delta_{\boldsymbol{z}} : V_{\mathcal{T}} \to \mathbb{R}$, for all $\boldsymbol{z} \in \mathcal{V}$, be the linear form defined by

$$\Delta_{\boldsymbol{z}} u = \frac{1}{|K_{\boldsymbol{z}}|}\sum_{\boldsymbol{y}\in\mathcal{V}} u(\boldsymbol{y})T_{\boldsymbol{z}\boldsymbol{y}} = \frac{1}{|K_{\boldsymbol{z}}|}\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(u(\boldsymbol{y}) - u(\boldsymbol{z})),\ \forall u \in V_{\mathcal{T}},\ \forall \boldsymbol{z} \in \mathcal{V}. \tag{7}$$

We then define $\Delta_{\mathcal{T}} : V_{\mathcal{T}} \to L^2(\Omega)$ by

$$\Delta_{\mathcal{T}} u(\boldsymbol{x}) = \sum_{\boldsymbol{z}\in\mathcal{V}} \Delta_{\boldsymbol{z}} u\ \mathbf{1}_{K_{\boldsymbol{z}}}(\boldsymbol{x}),\quad \text{for a.e. } \boldsymbol{x} \in \Omega,\ \forall u \in V_{\mathcal{T}}. \tag{8}$$

We define the discrete space

$$V_{\mathcal{T},0} = \{u \in V_{\mathcal{T}}, u = 0 \text{ on } \partial\Omega\}. \tag{9}$$

Then the scheme for the approximation of Problem (2) consists in finding

$$u \in V_{\mathcal{T},0};\ \forall v \in V_{\mathcal{T},0},\ \int_\Omega \Delta_{\mathcal{T}} u(\boldsymbol{x})\Delta_{\mathcal{T}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega (f(\boldsymbol{x})v(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x}) + \ell(\boldsymbol{x})\Delta_{\mathcal{T}} v(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}. \tag{10}$$

An important relation for the mathematical analysis is

$$\forall u, v \in V_{\mathcal{T}},\ \int_\Omega \nabla u(\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\sum_{\boldsymbol{z}\in\mathcal{V}} |K_{\boldsymbol{z}}|v(\boldsymbol{z})\Delta_{\boldsymbol{z}} u = -\int_\Omega P_{\mathcal{T}} v(\boldsymbol{x})\Delta_{\mathcal{T}} u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{11}$$

where we define the piecewise constant reconstruction of the elements of $V_{\mathcal{T}}$ by

$$P_{\mathcal{T}} v(\boldsymbol{x}) = \sum_{\boldsymbol{z}\in\mathcal{V}} v(\boldsymbol{z})\ \mathbf{1}_{K_{\boldsymbol{z}}}(\boldsymbol{x}),\quad \text{for a.e. } \boldsymbol{x} \in \Omega,\ \forall v \in V_{\mathcal{T}}. \tag{12}$$

3 Convergence analysis

Lemma 3.1 (Piecewise reconstruction) *Let us assume Hypotheses* (3)*. Let $\mathcal{T}$ be a conforming simplicial mesh of Ω. We then have the following inequality:*

$$\|v - P_{\mathcal{T}} v\|_{L^2(\Omega)} \le h_{\mathcal{T}} \|\nabla v\|_{L^2(\Omega)^d}, \ \forall v \in V_{\mathcal{T}}. \tag{13}$$

PROOF. Let $S \in \mathcal{T}$, $\boldsymbol{z} \in \mathcal{V}_S$ and $\boldsymbol{x} \in S$. We have $v(\boldsymbol{x}) - v(\boldsymbol{z}) = \nabla_S v \cdot (\boldsymbol{x} - \boldsymbol{z})$, denoting by $\nabla_S v$ the constant gradient of v in S. This leads to $|v(\boldsymbol{x}) - v(\boldsymbol{z})| \le |\nabla_S v| \ h_{\mathcal{T}}$. Therefore we get

$$\int_\Omega (v(\boldsymbol{x}) - P_{\mathcal{T}} v(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{z}\in\mathcal{V}} \sum_{S\in\mathcal{T}} \int_{K_{\boldsymbol{z}}\cap S} (v(\boldsymbol{x}) - v(\boldsymbol{z}))^2 \mathrm{d}\boldsymbol{x} \le h_{\mathcal{T}}^2 \int_\Omega |\nabla v(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x},$$

which gives (13). □

Lemma 3.2
Let us assume Hypotheses (3)*. Let $\mathcal{T}$ be a conforming simplicial mesh of Ω. Then the following inequalities hold:*

$$\|\nabla w\|_{L^2(\Omega)^d} \le 2 \ \mathrm{diam}(\Omega) \|\Delta_{\mathcal{T}} w\|_{L^2(\Omega)}, \quad \forall w \in V_{\mathcal{T},0}, \tag{14}$$

and

$$\|w\|_{L^2(\Omega)} \le 2 \ \mathrm{diam}(\Omega)^2 \|\Delta_{\mathcal{T}} w\|_{L^2(\Omega)}, \quad \forall w \in V_{\mathcal{T},0}. \tag{15}$$

PROOF. Setting $v = w$ in (11), we get

$$\int_\Omega |\nabla w(\boldsymbol{x})|^2 = -\int_\Omega P_{\mathcal{T}} w(\boldsymbol{x}) \Delta_{\mathcal{T}} w(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.$$

Hence, by using the Cauchy-Schwarz inequality, we have

$$\|\nabla w\|_{L^2(\Omega)^d}^2 \le \|P_{\mathcal{T}} w\|_{L^2(\Omega)} \|\Delta_{\mathcal{T}} w\|_{L^2(\Omega)}. \tag{16}$$

The Poincaré inequality [5], which holds since $V_{\mathcal{T},0} \subset H_0^1(\Omega)$, reads

$$\|w\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega) \|\nabla w\|_{L^2(\Omega)^d}.$$

By using (13) and $h_{\mathcal{T}} \le \mathrm{diam}(\Omega)$, we have

$$\begin{aligned} \|P_{\mathcal{T}} w\|_{L^2(\Omega)} \le \quad & \|P_{\mathcal{T}} w - w\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)} \le (h_{\mathcal{T}} + \mathrm{diam}(\Omega)) \|\nabla w\|_{L^2(\Omega)^d} \\ & \le 2 \ \mathrm{diam}(\Omega) \|\nabla w\|_{L^2(\Omega)^d}. \end{aligned}$$

Gathering the above results, we deduce (14) and (15) from (16).□

Lemma 3.3 (Existence, uniqueness and estimate on the solution of (10)**)** *Let us assume Hypotheses* (3) *and* (4)*. Let $\mathcal{T}$ be a conforming simplicial mesh of Ω. Then, for any $u \in V_{\mathcal{T},0}$ satisfying* (10)*, the following holds:*

$$\|u\|_{L^2(\Omega)} \le 4 \ \mathrm{diam}(\Omega)^4 \|f\|_{L^2(\Omega)} + 4 \ \mathrm{diam}(\Omega)^3 \|\boldsymbol{g}\|_{L^2(\Omega)^d} + 2 \ \mathrm{diam}(\Omega)^2 \|\ell\|_{L^2(\Omega)}, \tag{17}$$

$$\|\nabla u\|_{L^2(\Omega)^d} \le 4 \ \mathrm{diam}(\Omega)^3 \|f\|_{L^2(\Omega)} + 4 \ \mathrm{diam}(\Omega)^2 \|\boldsymbol{g}\|_{L^2(\Omega)^d} + 2 \ \mathrm{diam}(\Omega) \|\ell\|_{L^2(\Omega)}, \tag{18}$$

and

$$\|\Delta_{\mathcal{T}} u\|_{L^2(\Omega)} \le 2 \ \mathrm{diam}(\Omega)^2 \|f\|_{L^2(\Omega)} + 2 \ \mathrm{diam}(\Omega) \|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|\ell\|_{L^2(\Omega)}. \tag{19}$$

As a consequence, there exists one and only one $u \in V_{\mathcal{T},0}$ such that (10) *holds.*

PROOF. Let u be given such that (10) holds. Let us take $v = u$ in (10). We get

$$\int_\Omega (\Delta_{\mathcal{T}} u(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} = \int_\Omega (f(\boldsymbol{x})u(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x}) \cdot \nabla u(\boldsymbol{x}) + \ell(\boldsymbol{x})\Delta_{\mathcal{T}} u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x},$$

which leads to

$$\|\Delta_{\mathcal{T}} u\|_{L^2(\Omega)}^2 \le \|u\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)^d}\|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|\ell\|_{L^2(\Omega)})\|\Delta_{\mathcal{T}} u\|_{L^2(\Omega)},$$

Thanks to Lemma 3.2, the previous inequality provides

$$\|\Delta_{\mathcal{T}} u\|_{L^2(\Omega)} \le 2\ \mathrm{diam}(\Omega)^2\|f\|_{L^2(\Omega)} + 2\ \mathrm{diam}(\Omega)\|\boldsymbol{g}\|_{L^2(\Omega)^d} + \|\ell\|_{L^2(\Omega)},$$

which is (19). We then deduce (18) and (17), using again Lemma 3.2.
On the other hand, we remark that (10) is equivalent to a square linear system. Setting $f = 0$, $\boldsymbol{g} = 0$ and $\ell = 0$, we get from (17) that $u = 0$, showing the invertibility of the matrix of the system. This implies the existence and uniqueness of the discrete solution. $\square$

Lemma 3.4 (Compactness of a sequence of approximate solutions) *Let us assume Hypotheses (3). Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of conforming simplicial discretizations of Ω such that $h_{\mathcal{T}_m}$ tends to 0 as $m \to \infty$. Assume that there exists $\theta > 0$ with $\theta < \theta_{\mathcal{T}_m}$ for all $m \in \mathbb{N}$. Let $(u_m)_{m\in\mathbb{N}}$ be a sequence of functions such that $u_m \in V^0_{\mathcal{T}_m,0}$ for all $m \in \mathbb{N}$. For simplicity, we shall denote the discrete operator $\Delta_{\mathcal{T}_m}$ by Δ_m. Assume that the sequence $(\Delta_m u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega)$ by $\overline{C} \ge 0$; then there exists a subsequence of $(\mathcal{T}_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{T}_m)_{m\in\mathbb{N}}$, and $u \in H^2_0(\Omega)$, such that the corresponding subsequence $(u_m)_{m\in\mathbb{N}}$ satisfies:*

1. *$u_m \to u$ in $L^2(\Omega)$,*
2. *$\nabla u_m \to \nabla u$ in $L^2(\Omega)^d$,*
3. *$\Delta_m u_m \rightharpoonup \Delta u$ weakly in $L^2(\Omega)$,*

as $m \to \infty$.

PROOF. Since the sequence $(\Delta_m u_m)_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, we may extract a subsequence of $(\mathcal{T}_m, u_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{T}_m, u_m)_{m\in\mathbb{N}}$, such that $(\Delta_m u_m)_{m\in\mathbb{N}}$ converges weakly in $L^2(\Omega)$ to some $w \in L^2(\Omega)$. From Lemma 3.2, we get that

$$\|\nabla u_m\|_{L^2(\Omega)} \le C,\ \forall m \in \mathbb{N},$$

where $C \in \mathbb{R}_+$ only depends on Ω and $\overline{C}$. Therefore, applying Rellich's theorem, we get the existence of some $u \in H^1_0(\Omega)$ and of a subsequence of $(\mathcal{T}_m, u_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{T}_m, u_m)_{m\in\mathbb{N}}$, such that

$$\nabla u_m \rightharpoonup \nabla u \ \text{ weakly in } \quad L^2(\Omega)^d,$$

and

$$u_m \to u \ \text{ strongly in } \quad L^2(\Omega),$$

as $m \to \infty$. Let us prove that $u \in H^2_0(\Omega)$. Let $\overline{u}$ (resp. $\overline{u}_m$) denote the prolongement of u (resp. u_m) by 0 in $\mathbb{R}^d \setminus \Omega$. Thanks to $u \in H^1_0(\Omega)$ (resp. $u_m \in H^1_0(\Omega)$), we have $\nabla\overline{u} \in L^2(\mathbb{R}^d)^d$ (resp. $\nabla\overline{u}_m \in L^2(\mathbb{R}^d)^d$), with the property

$$\nabla\overline{u}_m \rightharpoonup \nabla\overline{u} \ \text{ weakly in } \quad L^2(\mathbb{R}^d)^d. \tag{20}$$

Let $\varphi \in C^\infty_c(\mathbb{R}^d)$; note that φ does not necessarily vanish at the boundary of Ω. Let $\mathcal{I}_m\varphi$ denote $\mathcal{I}_{\mathcal{T}_m}\varphi$ for short. Recall that $\mathcal{I}_m\varphi$ tends to φ in $H^1(\Omega)$ as $m \to \infty$, thanks to the hypothesis that there exists $\theta > 0$ with $\theta < \theta_{\mathcal{T}_m}$ for all $m \in \mathbb{N}$. We then define the approximation $\boldsymbol{G}_m\varphi$ of $\nabla\varphi$ by

$$\boldsymbol{G}_m\varphi(\boldsymbol{x}) = \begin{cases} \nabla\mathcal{I}_m\varphi(\boldsymbol{x}) & \text{for a.e} \quad \boldsymbol{x} \in \Omega, \\ \nabla\varphi(\boldsymbol{x}) & \text{for a.e} \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \Omega. \end{cases} \tag{21}$$

Let $T_m = \int_{\mathbb{R}^d} \nabla \overline{u}_m(\boldsymbol{x}) \cdot \boldsymbol{G}_m\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$. Using (20), and the convergence of $\boldsymbol{G}_m\varphi(\boldsymbol{x})$ to $\nabla\varphi$ in $L^2(\mathbb{R}^d)^d$, we get

$$\lim_{m\to+\infty} T_m = \int_{\mathbb{R}^d} \nabla \overline{u}(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

On the other hand, we have

$$T_m = \int_{\mathbb{R}^d} \nabla \overline{u}_m(\boldsymbol{x}) \cdot \boldsymbol{G}_m\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\Omega} \nabla u_m(\boldsymbol{x}) \cdot \nabla\mathcal{I}_m\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Thanks to (11), we get

$$T_m = -\int_{\Omega} P_{\mathcal{T}_m}\mathcal{I}_m\varphi(\boldsymbol{x})\Delta_m u(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Passing to the limit $m \to \infty$ in the above relation, since (13) shows that $P_{\mathcal{T}_m}\mathcal{I}_m\varphi$ converges to φ in $L^2(\Omega)$, we get thanks to strong/weak convergence properties,

$$\int_{\mathbb{R}^d} \nabla \overline{u}(\boldsymbol{x}) \cdot \nabla\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\int_{\Omega} \varphi(\boldsymbol{x})w(\boldsymbol{x}) = -\int_{\mathbb{R}^d} \varphi(\boldsymbol{x})\overline{w}(\boldsymbol{x}),$$

where we denote by $\overline{w}$ the prolongement of w by 0 in $\mathbb{R}^d \setminus \Omega$. This proves that $\nabla\overline{u} \in H_{\mathrm{div}}(\mathbb{R}^d)$ and that $\Delta\overline{u} = \overline{w}$ a.e. in $\mathbb{R}^d$, which means that $\Delta\overline{u} = 0$ outside Ω and that $\Delta u = w$ a.e. in Ω. Since $\overline{u} \in H^1(\mathbb{R}^d)$ and $\Delta\overline{u} \in L^2(\mathbb{R}^d)$, a classical result of regularity shows that $\overline{u} \in H^2(\mathbb{R}^d)$. Since $\nabla\overline{u} = 0$ in $\mathbb{R}^d \setminus \Omega$, we get that the trace of ∇u on $\partial\Omega$ is equal to 0. Hence $u \in H_0^2(\Omega)$.
Let us now prove the strong convergence of ∇u_m to ∇u. Using the weak convergence of this sequence, it suffices to prove the convergence of $\|\nabla u_m\|_{L^2(\Omega)^d}$ to $\|\nabla u\|_{L^2(\Omega)^d}$. To this aim, we write the relation obtained by setting $u = v = u_m$ in (11):

$$\int_{\Omega} |\nabla u_m(\boldsymbol{x})|^2 = -\int_{\Omega} P_{\mathcal{T}_m}u_m(\boldsymbol{x})\Delta_m u_m(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\ \forall m \in \mathbb{N}.$$

Passing to the limit $m \to \infty$ in the above relation, we get, using strong/weak convergence properties in the right hand side,

$$\lim_{m\to\infty} \int_{\Omega} |\nabla u_m(\boldsymbol{x})|^2 = -\int_{\Omega} u(\boldsymbol{x})\Delta u(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\Omega} |\nabla u(\boldsymbol{x})|^2,$$

hence concluding the proof.□
In order to conclude the convergence analysis, it is natural to examine the convergence of the discrete Laplace operator, when applied to $\mathcal{I}_{\mathcal{T}}\varphi \in V_{\mathcal{T},0}$, for any regular function $\varphi \in C^2(\overline{\Omega}) \cap H_0^2(\Omega)$. We show in the next lemma that this operator is indeed a second order discrete operator.

Lemma 3.5 (Order of the discrete Laplace operator applied to standard interpolation) *Let us assume Hypotheses* (3). *Let $\mathcal{T}$ be a conforming simplicial mesh of Ω and let $\theta > 0$ such that $\theta \le \theta_{\mathcal{T}}$. Then there exists $C_1 > 0$, only depending on θ, such that*

$$|\nabla_S \xi_{\boldsymbol{z}}| \le \frac{C_1}{h_S},\ \forall \boldsymbol{z} \in \mathcal{V}_S, \forall S \in \mathcal{T}, \tag{22}$$

and

$$|\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\varphi| \le C_1|\varphi|_2,\ \forall\varphi \in C^2(\overline{\Omega}),\ \forall \boldsymbol{z} \in \mathcal{V}_{\mathrm{int}}, \tag{23}$$

where $\Delta_{\boldsymbol{z}}$ is defined by (7), *$\mathcal{I}_{\mathcal{T}}\varphi$ is defined by* (5) *and $|\varphi|_2 = \max_{i,j=1,d} \|\partial_{ij}^2\varphi\|_{L^\infty(\Omega)}$.*

Proof. Inequality (22) results from the fact that the ratio, between the distance from any vertex of S to the opposite face and h_S, is larger than 2θ. We now consider $\boldsymbol{z} \in \mathcal{V}_{\mathrm{int}}$ and $\varphi \in C^2(\overline{\Omega})$. We can write, using (7),

$$\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\varphi = \frac{1}{|K_{\boldsymbol{z}}|} \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} (\varphi(\boldsymbol{y}) - \varphi(\boldsymbol{z}))T_{\boldsymbol{z}\boldsymbol{y}}.$$

A Taylor expansion provides $\varphi(\boldsymbol{x}) - \varphi(\boldsymbol{z}) = \boldsymbol{G}\cdot(\boldsymbol{x}-\boldsymbol{z}) + D(\boldsymbol{x},\boldsymbol{z})|\boldsymbol{x}-\boldsymbol{z}|^2$, where $|D(\boldsymbol{x},\boldsymbol{z})| \le d^2|\varphi|_2$ and $\boldsymbol{G} = \nabla\varphi(\boldsymbol{z})$. Let us check that the discrete operator $\Delta_{\boldsymbol{z}}$ vanishes on the affine function $\mu\ :\ \boldsymbol{x}\mapsto \boldsymbol{G}\cdot(\boldsymbol{x}-\boldsymbol{z})$ (which is such that $\mu\in V_{\mathcal{T}}$). Indeed, we have, using (11) and (12),

$$\begin{aligned}-|K_{\boldsymbol{z}}|\Delta_{\boldsymbol{z}}\mu &= -\int_\Omega \Delta_{\mathcal{T}}\mu(\boldsymbol{x})\mathbf{1}_{K_{\boldsymbol{z}}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\int_\Omega \Delta_{\mathcal{T}}\mu(\boldsymbol{x})P_{\mathcal{T}}\xi_{\boldsymbol{z}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\\ &= \int_\Omega \nabla\mu(\boldsymbol{x})\cdot\nabla\xi_{\boldsymbol{z}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{G}\cdot\nabla\xi_{\boldsymbol{z}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0.\end{aligned}$$

We therefore get

$$\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\varphi = \frac{1}{|K_{\boldsymbol{z}}|}\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{zy}}D(\boldsymbol{y},\boldsymbol{z})|\boldsymbol{y}-\boldsymbol{z}|^2 = -\frac{1}{|K_{\boldsymbol{z}}|}\sum_{S\in\mathcal{T}_{\boldsymbol{z}}}\sum_{\boldsymbol{y}\in\mathcal{V}_S}|S|\nabla_S\xi_{\boldsymbol{z}}\cdot\nabla_S\xi_{\boldsymbol{y}}D(\boldsymbol{y},\boldsymbol{z})|\boldsymbol{y}-\boldsymbol{z}|^2.$$

Using (22) and the regularity condition $\theta\sum_{S\in\mathcal{T}_{\boldsymbol{z}}}|S| \le\ |K_{\boldsymbol{z}}|$ and $|\boldsymbol{y}-\boldsymbol{z}|\le h_S$, we conclude (23). □
Since the discrete Laplace operator is a second order discrete operator, the question of its strong convergence to the continuous Laplace operator arises. Indeed, the proof that $\Delta_{\mathcal{T}}\mathcal{I}_{\mathcal{T}}\varphi$ converges to $\Delta\varphi$ for the weak topology of $L^2(\Omega)$ results from the following property, the proof of which uses Lemma 3.1:

$$\int_\Omega(\Delta_{\mathcal{T}}\mathcal{I}_{\mathcal{T}}\varphi - \Delta\varphi)P_{\mathcal{T}}v\mathrm{d}\boldsymbol{x} = \int_\Omega \Delta\varphi(v-P_{\mathcal{T}}v)\mathrm{d}\boldsymbol{x} - \int_\Omega(\nabla\mathcal{I}_{\mathcal{T}}\varphi-\nabla\varphi)\cdot\nabla v\mathrm{d}\boldsymbol{x} \le Ch_{\mathcal{T}}\|\nabla v\|_{L^2(\Omega)^d},\ \forall v\in V_{\mathcal{T},0}.$$

Nevertheless, whatever be the choice of $(K_{\boldsymbol{z}})_{\boldsymbol{z}\in\mathcal{V}}$ satisfying the hypotheses required above, it is not possible in the general case to obtain that $\Delta_{\mathcal{T}}\mathcal{I}_{\mathcal{T}}\varphi$ strongly converges to $\Delta\varphi$ as $h_{\mathcal{T}}\to 0$, while $\theta\le\theta_{\mathcal{T}}$. Therefore it is not possible to conclude to the convergence of the scheme by letting $v = \mathcal{I}_{\mathcal{T}}\varphi$ in (10): another interpolation is necessary, which we introduce in the following Lemma 3.6.

Lemma 3.6 (Interpolation of regular functions with compact support) *Let us assume Hypotheses (3). Let $\mathcal{T}$ be a conforming simplicial discretization of Ω, and let $\theta>0$ be given such that $\theta<\theta_{\mathcal{T}}$. Let $\varphi\in C_c^2(\Omega)$ and let $a = \mathrm{d}(\mathrm{support}(\varphi),\partial\Omega)$.*
Then there exists $\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi\in V_{\mathcal{T},0}$ and $C>0$ only depending on Ω and θ such that

$$\|\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi - \varphi\|_{L^2(\Omega)} \le Ch_{\mathcal{T}}\frac{|\varphi|_2}{a^2}, \tag{24}$$

$$\|\nabla\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi - \nabla\varphi\|_{L^2(\Omega)^d} \le Ch_{\mathcal{T}}\frac{|\varphi|_2}{a^2}, \tag{25}$$

and

$$\|\Delta_{\mathcal{T}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi - \overline{\Delta}_{\mathcal{T}}\varphi\|_{L^2(\Omega)} \le Ch_{\mathcal{T}}\frac{|\varphi|_2}{a^2}, \tag{26}$$

where $|\varphi|_2 = \max_{i,j=1,d}\|\partial^2_{ij}\varphi\|_{L^\infty(\Omega)}$ and $\overline{\Delta}_{\mathcal{T}}\varphi$ is the piecewise constant function defined by

$$\overline{\Delta}_{\boldsymbol{z}}\varphi = \frac{1}{|K_{\boldsymbol{z}}|}\int_{K_{\boldsymbol{z}}}\Delta\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\ \forall\boldsymbol{z}\in\mathcal{V}, \tag{27}$$

and

$$\overline{\Delta}_{\mathcal{T}}\varphi(\boldsymbol{x}) = \sum_{\boldsymbol{z}\in\mathcal{V}}\overline{\Delta}_{\boldsymbol{z}}\varphi\ \mathbf{1}_{K_{\boldsymbol{z}}}(\boldsymbol{x}),\quad \textit{for a.e. } \boldsymbol{x}\in\Omega. \tag{28}$$

Proof. Let $\rho\in C_c^\infty(\mathbb{R}^d,\mathbb{R}_+)$ be the function defined by

$$\rho(\boldsymbol{x}) = \frac{\exp(-1/(1-|\boldsymbol{x}|^2))}{\int_{B(0,1)}\exp(-1/(1-|\boldsymbol{y}|^2))\mathrm{d}\boldsymbol{y}},\ \forall\boldsymbol{x}\in B(0,1),$$

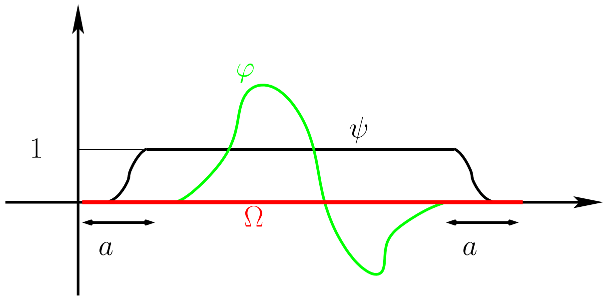


Figure 2: Functions φ and ψ

and $\rho(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin B(0,1)$. Let $\psi \in C_c^\infty(\Omega, [0,1])$ be the function defined by

$$\psi(\boldsymbol{y}) = \int_{\boldsymbol{x}\in\Omega, \mathrm{d}(\boldsymbol{x},\partial\Omega) > \frac{a}{2}} \left(\frac{4}{a}\right)^d \rho\left(\frac{4}{a}(\boldsymbol{y}-\boldsymbol{x})\right) \mathrm{d}\boldsymbol{x}, \ \forall \boldsymbol{y} \in \Omega. \tag{29}$$

Then $\psi(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \Omega$ such that $\mathrm{d}(\boldsymbol{x}, \partial\Omega) < \frac{a}{4}$ and $\psi(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \Omega$ such that $\mathrm{d}(\boldsymbol{x}, \partial\Omega) > \frac{3a}{4}$ (see Figure 2). The idea of the construction of $\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi$ is to consider the approximation of φ in $V_{\mathcal{T},0}$ obtained by the finite element method in the case where the right hand side is given by $-\Delta\varphi$; since $\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi$ must be equal to 0 on the boundary cells, we multiply this discrete solution by ψ. Then the proof mimics the identity $\Delta(\psi v) = v\Delta\psi + 2\nabla\psi\cdot\nabla v + \psi\Delta v$.

We first suppose that $\mathcal{T}$ is such that $h_{\mathcal{T}} < \frac{a}{4}$. Let us define $\widehat{v}$ and $\widetilde{v} \in V_{\mathcal{T},0}$ such that

$$\forall v \in V_{\mathcal{T},0}, \begin{cases} \displaystyle\int_\Omega \nabla\widetilde{v}(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\int_\Omega \Delta\varphi(\boldsymbol{x}) P_{\mathcal{T}} v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \\ \text{and } \displaystyle\int_\Omega \nabla\widehat{v}(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = -\int_\Omega \Delta\varphi(\boldsymbol{x}) v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \end{cases} \tag{30}$$

We define $\widetilde{w} = \widetilde{v} - \widehat{v}$. By subtracting the second relation to the first one in (30), and setting $v = \widetilde{w}$, we get

$$\int_\Omega |\nabla\widetilde{w}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = -\int_\Omega \Delta\varphi(\boldsymbol{x})(P_{\mathcal{T}}\widetilde{w}(\boldsymbol{x}) - \widetilde{w}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}.$$

Applying Lemma 3.1, we obtain

$$\int_\Omega |\nabla\widetilde{w}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \le \|\Delta\varphi\|_{L^2(\Omega)} \|P_{\mathcal{T}}\widetilde{w}(\boldsymbol{x}) - \widetilde{w}(\boldsymbol{x})\|_{L^2(\Omega)} \le h_{\mathcal{T}} \|\Delta\varphi\|_{L^2(\Omega)} \|\nabla\widetilde{w}\|_{L^2(\Omega)^d}.$$

We then deduce

$$\|\nabla\widetilde{v} - \nabla\widehat{v}\|_{L^2(\Omega)^d} \le h_{\mathcal{T}} \|\Delta\varphi\|_{L^2(\Omega)}.$$

A standard interpolation result gives the existence of C_θ, only depending on Ω and θ, such that

$$\|\nabla\varphi - \nabla\mathcal{I}_{\mathcal{T}}\varphi\|_{L^2(\Omega)} \le C_\theta h_{\mathcal{T}} |\varphi|_2. \tag{31}$$

Thanks to

$$\int_\Omega |\nabla\widehat{v}(\boldsymbol{x}) - \nabla\varphi(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \int_\Omega (\nabla\widehat{v}(\boldsymbol{x}) - \nabla\varphi(\boldsymbol{x}))\cdot(\nabla\mathcal{I}_{\mathcal{T}}\varphi(\boldsymbol{x}) - \nabla\varphi(\boldsymbol{x}))\mathrm{d}\boldsymbol{x},$$

we also get

$$\|\nabla\widehat{v} - \nabla\varphi\|_{L^2(\Omega)^d} \le C_\theta h_{\mathcal{T}} |\varphi|_2.$$

Hence, defining $w = \widetilde{v} - \mathcal{I}_{\mathcal{T}}\varphi$, we obtain

$$\|\nabla w\|_{L^2(\Omega)^d} \le \|\nabla(\widetilde{v} - \widehat{v})\|_{L^2(\Omega)} + \|\nabla(\widehat{v} - \varphi)\|_{L^2(\Omega)} + \|\nabla(\varphi - \mathcal{I}_{\mathcal{T}}\varphi)\|_{L^2(\Omega)} \le (2C_\theta + 1) h_{\mathcal{T}} |\varphi|_2. \tag{32}$$

The Poincaré inequality, which holds since $w \in H_0^1(\Omega)$,writes

$$\|w\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega)\|\nabla w\|_{L^2(\Omega)^d} \leq \mathrm{diam}(\Omega)(2C_\theta + 1)h_{\mathcal{T}}|\varphi|_2. \tag{33}$$

Let us remark that, thanks to (30), $\widetilde{v}$ satisfies

$$\Delta_{\boldsymbol{z}}\widetilde{v} = \overline{\Delta}_{\boldsymbol{z}}\varphi,\ \forall \boldsymbol{z} \in \mathcal{V}_{\mathrm{int}}, \tag{34}$$

using the notation given by (27) (note that this equality is not a priori satisfied for $\boldsymbol{z} \in \mathcal{V}_{\mathrm{ext}}$). We now observe that we have

$$\psi(\boldsymbol{z})\Delta_{\boldsymbol{z}}\widetilde{v} = \overline{\Delta}_{\boldsymbol{z}}\varphi,\ \forall \boldsymbol{z} \in \mathcal{V}. \tag{35}$$

Indeed, if $\mathrm{d}(\boldsymbol{z}, \partial\Omega) > \frac{3a}{4}$, we have $\psi(\boldsymbol{z}) = 1$ and $\boldsymbol{z} \in \mathcal{V}_{\mathrm{int}}$, which implies that (34) holds. Otherwise, if $\mathrm{d}(\boldsymbol{z}, \partial\Omega) \leq \frac{3a}{4}$ and $\boldsymbol{z} \in \mathcal{V}_{\mathrm{int}}$, we have $|K_{\boldsymbol{z}}|\Delta_{\boldsymbol{z}}\widetilde{v} = \int_{K_{\boldsymbol{z}}} \Delta\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$, and if $\boldsymbol{z} \in \mathcal{V}_{\mathrm{ext}}$, we have $\psi(\boldsymbol{z}) = 0$ and $\int_{K_{\boldsymbol{z}}} \Delta\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0$. We now define $\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi \in V_{\mathcal{T},0}$ by $\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi(\boldsymbol{z}) = \psi(\boldsymbol{z})\widetilde{v}(\boldsymbol{z})$, for all $\boldsymbol{z} \in \mathcal{V}$. From formula (7), we get

$$|K_{\boldsymbol{z}}|\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi = \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi(\boldsymbol{y}) - \widetilde{\mathcal{I}}_{\mathcal{T}}\varphi(\boldsymbol{z})) = \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\psi(\boldsymbol{y})\widetilde{v}(\boldsymbol{y}) - \psi(\boldsymbol{z})\widetilde{v}(\boldsymbol{z})).$$

Thanks to the identity $ab - cd = c(b-d) + d(a-c) + (a-c)(b-d)$, we get, for $\boldsymbol{z}, \boldsymbol{y} \in \mathcal{V}$,

$$\psi(\boldsymbol{y})\widetilde{v}(\boldsymbol{y}) - \psi(\boldsymbol{z})\widetilde{v}(\boldsymbol{z}) = \psi(\boldsymbol{z})(\widetilde{v}(\boldsymbol{y}) - \widetilde{v}(\boldsymbol{z})) + \widetilde{v}(\boldsymbol{z})(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z})) + (\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))(\widetilde{v}(\boldsymbol{y}) - \widetilde{v}(\boldsymbol{z})).$$

This implies, from formula (7), that

$$|K_{\boldsymbol{z}}|\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi = \psi(\boldsymbol{z})\ |K_{\boldsymbol{z}}|\ \Delta_{\boldsymbol{z}}\widetilde{v} + \widetilde{v}(\boldsymbol{z})\ |K_{\boldsymbol{z}}|\ \Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi + \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))(\widetilde{v}(\boldsymbol{y}) - \widetilde{v}(\boldsymbol{z})). \tag{36}$$

We remark that

$$(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))\varphi(\boldsymbol{y}) = (\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))\varphi(\boldsymbol{z}) = 0,\ \forall \boldsymbol{z} \in \mathcal{V},\ \forall \boldsymbol{y} \in \mathcal{V}_{\boldsymbol{z}}. \tag{37}$$

Indeed, assuming $\varphi(\boldsymbol{y}) \neq 0$ or $\varphi(\boldsymbol{z}) \neq 0$, we have $\mathrm{d}(\boldsymbol{z}, \partial\Omega) > a$ or $\mathrm{d}(\boldsymbol{y}, \partial\Omega) > a$. Since $\mathrm{d}(\boldsymbol{z}, \boldsymbol{y}) \leq h_{\mathcal{T}} \leq a/4$, we get that $\mathrm{d}(\boldsymbol{z}, \partial\Omega) > \frac{3a}{4}$ and $\mathrm{d}(\boldsymbol{y}, \partial\Omega) > \frac{3a}{4}$. This implies $\psi(\boldsymbol{z}) = 1$ and $\psi(\boldsymbol{y}) = 1$. Therefore $\psi(\boldsymbol{z}) - \psi(\boldsymbol{y}) = 0$.

We then get from (37), for all $\boldsymbol{z} \in \mathcal{V}$ and $\boldsymbol{y} \in \mathcal{V}_{\boldsymbol{z}}$,

$$(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))(\widetilde{v}(\boldsymbol{y}) - \widetilde{v}(\boldsymbol{z})) = (\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))(\widetilde{v}(\boldsymbol{y}) - \varphi(\boldsymbol{y}) - \widetilde{v}(\boldsymbol{z}) + \varphi(\boldsymbol{z}))$$

and

$$\widetilde{v}(\boldsymbol{z})\ \Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi = \widetilde{v}(\boldsymbol{z}) \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z})) = (\widetilde{v}(\boldsymbol{z}) - \varphi(\boldsymbol{z})) \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z})) = w(\boldsymbol{z})\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi.$$

Using (35) and the two preceding relations in (36), we obtain

$$\begin{aligned} &|K_{\boldsymbol{z}}|\left(\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi - \overline{\Delta}_{\boldsymbol{z}}\varphi\right) \\ &= w(\boldsymbol{z})|K_{\boldsymbol{z}}|\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi + \sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))(w(\boldsymbol{y}) - w(\boldsymbol{z})). \end{aligned}$$

Taking the square of the previous relation and applying the inequality $(a+b)^2 \leq 2(a^2+b^2)$ we get,

$$\begin{aligned} |K_{\boldsymbol{z}}|^2\left(\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi - \overline{\Delta}_{\boldsymbol{z}}\varphi\right)^2 \leq\ & 2\left(w(\boldsymbol{z})|K_{\boldsymbol{z}}|\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi\right)^2 \\ & +2\left(\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}} T_{\boldsymbol{z}\boldsymbol{y}}(\psi(\boldsymbol{y}) - \psi(\boldsymbol{z}))(w(\boldsymbol{y}) - w(\boldsymbol{z}))\right)^2. \end{aligned}$$

Using the Cauchy-Schwarz inequality and dividing by $|K_{\boldsymbol{z}}|$, we obtain

$$\begin{aligned}|K_{\boldsymbol{z}}|\left(\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi-\overline{\Delta}_{\boldsymbol{z}}\varphi\right)^2 \quad &\le 2\ w(\boldsymbol{z})^2|K_{\boldsymbol{z}}|(\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi)^2\\ &+\frac{2}{|K_{\boldsymbol{z}}|}\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}}|T_{\boldsymbol{z}\boldsymbol{y}}|(\psi(\boldsymbol{y})-\psi(\boldsymbol{z}))^2\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}}|T_{\boldsymbol{z}\boldsymbol{y}}|(w(\boldsymbol{y})-w(\boldsymbol{z}))^2.\end{aligned}$$

We now use (22), which implies that

$$\left|\int_S\nabla\xi_{\boldsymbol{z}}(\boldsymbol{x})\cdot\nabla\xi_{\boldsymbol{y}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right|\le\frac{|S|C_1^2}{h_S^2},\ \forall\boldsymbol{y},\boldsymbol{z}\in\mathcal{V}_S,\ \forall S\in\mathcal{T}.$$

Thanks to the definition (29) of ψ, we have the existence of a constant C_3 such that

$$\|\nabla\psi\|\le\frac{C_3}{a}.$$

This leads, using $|\psi(\boldsymbol{y})-\psi(\boldsymbol{z})|\le\frac{C_3}{a}h_S$, to the existence of C_4, only depending on θ, such that

$$\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}}|T_{\boldsymbol{z}\boldsymbol{y}}|(\psi(\boldsymbol{y})-\psi(\boldsymbol{z}))^2\le\frac{C_4}{a^2}|K_{\boldsymbol{z}}|.$$

Applying (23) proved in Lemma 3.5, since $|\psi|_2\le C_5/a^2$,

$$|\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi|\le\frac{C_6}{a^2},$$

where C_6 only depends on θ (this inequality also holds for $\boldsymbol{z}\in\mathcal{V}_{\rm ext}$ since in this case $\Delta_{\boldsymbol{z}}\mathcal{I}_{\mathcal{T}}\psi=0$). Hence we get

$$\begin{aligned}\sum_{\boldsymbol{z}\in\mathcal{V}}|K_{\boldsymbol{z}}|\left(\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi-\overline{\Delta}_{\boldsymbol{z}}\varphi\right)^2\le\quad &2\frac{C_6^2}{a^4}\sum_{\boldsymbol{z}\in\mathcal{V}}|K_{\boldsymbol{z}}|w(\boldsymbol{z})^2\\ &+2\frac{C_4}{a^2}\sum_{\boldsymbol{z}\in\mathcal{V}}\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}}|T_{\boldsymbol{z}\boldsymbol{y}}|(w(\boldsymbol{y})-w(\boldsymbol{z}))^2.\end{aligned}$$

We now remark that, for $S\in\mathcal{T}$ such that $\boldsymbol{y},\boldsymbol{z}\in\mathcal{V}_S$, we have

$$|T_{\boldsymbol{z}\boldsymbol{y}}|(w(\boldsymbol{y})-w(\boldsymbol{z}))^2\le\sum_{S\in\mathcal{T},\boldsymbol{y},\boldsymbol{z}\in\mathcal{V}_S}|T^S_{\boldsymbol{z}\boldsymbol{y}}|(\nabla_S w\cdot(\boldsymbol{y}-\boldsymbol{z}))^2\le C_1^2\sum_{S\in\mathcal{T},\boldsymbol{y},\boldsymbol{z}\in\mathcal{V}_S}\frac{|S|}{h_S^2}|\nabla_S w|^2h_S^2,$$

where we denote by $T^S_{\boldsymbol{z}\boldsymbol{y}}=-|S|\nabla_S\xi_{\boldsymbol{z}}\cdot\nabla_S\xi_{\boldsymbol{y}}$. Hence, we may write

$$\sum_{\boldsymbol{z}\in\mathcal{V}}\sum_{\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{z}}}|T_{\boldsymbol{z}\boldsymbol{y}}|(w(\boldsymbol{y})-w(\boldsymbol{z}))^2\le 2\frac{d(d+1)C_1^2}{2}\sum_{S\in\mathcal{T}}|S|\ |\nabla_S w|^2$$

remarking that each edge of a simplex occurs two times in the above summation, and that any simplex has $\frac{d(d+1)}{2}$ edges. Gathering the above results, we thus obtain

$$\sum_{\boldsymbol{z}\in\mathcal{V}}|K_{\boldsymbol{z}}|\left(\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi-\overline{\Delta}_{\boldsymbol{z}}\varphi\right)^2\le\quad 2\frac{C_6^2}{a^4}\|w\|^2_{L^2(\Omega)}+2d(d+1)\frac{C_4\,C_1^2}{a^2}\|\nabla w\|^2_{L^2(\Omega)^d}.$$

Using (33) and (32) provides (26). We then remark that, for all $v\in V_{\mathcal{T},0}$, we have

$$\int_\Omega(\Delta_{\boldsymbol{z}}\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi(\boldsymbol{x})-\overline{\Delta}_{\boldsymbol{z}}\varphi(\boldsymbol{x}))P_{\mathcal{T}}v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=-\int_\Omega\nabla\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}+\int_\Omega\nabla\widetilde{v}(\boldsymbol{x})\cdot\nabla v(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

thanks to both (11) and (30). Hence taking $v=\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi-\widetilde{v}$ provides (25), as well as (24) using the Poincaré inequality.
The proof of (26) in the case $h_{\mathcal{T}}\ge\frac{a}{4}$ is obtained by defining $\widetilde{\mathcal{I}}_{\mathcal{T}}\varphi=0$, and using $\|\overline{\Delta}_{\mathcal{T}}\varphi\|_{L^2(\Omega)}\le|\varphi|_2|\Omega|^{1/2}$, $h_{\mathcal{T}}/a\ge 1/4$ and $1/a\ge 1/\mathrm{diam}(\Omega)$. Then (25) and (24) follow in that case. □

Theorem 3.7 (Convergence of the scheme) *Let us assume Hypotheses (3) and (4). Let $u \in H_0^2(\Omega)$ be the solution of Problem (2); let $\mathcal{T}$ be a conforming simplicial mesh of Ω and $u_{\mathcal{T}} \in V_{\mathcal{T},0}$ be the solution of (10). Then, as $h_{\mathcal{T}}$ tends to 0 with $\theta \le \theta_{\mathcal{T}}$, for a fixed value of $\theta > 0$:*

1. *$u_{\mathcal{T}}$ converges in $L^2(\Omega)$ to u,*
2. *$\nabla u_{\mathcal{T}}$ converges in $L^2(\Omega)^d$ to ∇u,*
3. *$\Delta_{\mathcal{T}} u_{\mathcal{T}}$ converges in $L^2(\Omega)$ to Δu.*

Proof. Let $(\mathcal{T}_m)_{m\in\mathbb{N}}$ be a sequence of conforming simplicial meshes of Ω such that $h_{\mathcal{T}_m}$ tends to 0 as $m \to \infty$ and $\theta < \theta_{\mathcal{T}_m}$ for all $m \in \mathbb{N}$. Let $u_m \in V^0_{\mathcal{T}_m,0}$, for all $m \in \mathbb{N}$, be the solution of (10). Thanks to Lemmas 3.3 and 3.4, we get the existence of a subsequence of $(\mathcal{T}_m)_{m\in\mathbb{N}}$, again denoted $(\mathcal{T}_m)_{m\in\mathbb{N}}$, and of $u \in H_0^2(\Omega)$ such that the conclusion of Lemma 3.4 holds. Let $\varphi \in C_c^\infty(\Omega)$ be given. We take, in (10) with $\mathcal{T} = \mathcal{T}_m$, $v = \widetilde{\mathcal{I}}_{\mathcal{T}_m}\varphi$ defined by Lemma 3.6. Passing to the limit as $m \to \infty$ in the resulting equation (thanks to the weak/strong convergence properties provided by Lemmas 3.4 and 3.6) and using the density of $C_c^\infty(\Omega)$ in $H_0^2(\Omega)$, we get that u is the solution of Problem (2). By a classical uniqueness argument, we get that the whole sequence converges. Setting $v = u_m$ in (10), we get that $\|\Delta_{\mathcal{T}_m} u_m\|^2_{L^2(\Omega)}$ converges to $\int_\Omega (f(\boldsymbol{x})u(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})\cdot\nabla u(\boldsymbol{x}) + \ell(\boldsymbol{x})\Delta u(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x} = \int_\Omega (\Delta u(\boldsymbol{x}))^2\,\mathrm{d}\boldsymbol{x}$ as $m \to \infty$. Together with the weak convergence of $\Delta_{\mathcal{T}_m} u_m$ to Δu as $m \to \infty$, this provides the convergence in $L^2(\Omega)$ of $\Delta_{\mathcal{T}_m} u_m$ to Δu. $\square$

4 Error estimates

Let us first prove a technical lemma used in the following error estimate results.

Lemma 4.1 (An inequality for regular continuous solutions.)
Under Hypotheses (3), let $u \in C^4(\overline{\Omega})$ be given and let $f = \Delta(\Delta u)$. Let $\mathcal{T}$ be a conforming simplicial mesh of Ω and let $\theta > 0$ be such that $\theta < \theta_{\mathcal{T}}$. Let $u_{\mathcal{T}} \in V_{\mathcal{T},0}$ be the solution of (10) in the case where $f = \Delta(\Delta u)$, $\boldsymbol{g} = 0$ and $\ell = 0$. Then there exists $C > 0$, only depending on Ω and θ, such that the following inequality holds:

$$\|\Delta u - \Delta_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2(\Omega)} \le C h_{\mathcal{T}} \|u\|_{\infty,4} + 2\, \|\overline{\Delta}_{\mathcal{T}} u - \Delta_{\mathcal{T}} v\|_{L^2(\Omega)},\ \forall v \in V_{\mathcal{T},0}, \tag{38}$$

where, for all $k \in \mathbb{N}$, $\|u\|_{\infty,k}$ denotes the maximum value of the absolute value of u and of its derivatives until order k and $\overline{\Delta}_{\mathcal{T}}$ is defined by (28).

Proof. Let $v, w \in V_{\mathcal{T},0}$ be given. We have

$$\begin{aligned}\int_\Omega (\Delta_{\mathcal{T}} v(\boldsymbol{x}) - \Delta_{\mathcal{T}} u_{\mathcal{T}}(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = &\int_\Omega (\Delta_{\mathcal{T}} v(\boldsymbol{x}) - \overline{\Delta}_{\mathcal{T}} u(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &+ \int_\Omega (\overline{\Delta}_{\mathcal{T}} u(\boldsymbol{x}) - P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &+ \int_\Omega (P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)(\boldsymbol{x}) - \Delta_{\mathcal{T}} u_{\mathcal{T}}(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Using (10), $f = \Delta(\Delta u)$ and $w \in H_0^1(\Omega)$, we may write

$$\int_\Omega \Delta_{\mathcal{T}} u_{\mathcal{T}}(\boldsymbol{x})\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \int_\Omega \Delta(\Delta u)(\boldsymbol{x})\ w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla(\Delta u)(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

We have, from (11),

$$\int_\Omega P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)(\boldsymbol{x})\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \nabla(\mathcal{I}_{\mathcal{T}}\Delta u)(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Hence we get

$$\begin{aligned}\int_\Omega (\Delta_{\mathcal{T}} v(\boldsymbol{x}) - \Delta_{\mathcal{T}} u_{\mathcal{T}}(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = &\int_\Omega (\Delta_{\mathcal{T}} v(\boldsymbol{x}) - \overline{\Delta}_{\mathcal{T}} u(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &+\int_\Omega (\overline{\Delta}_{\mathcal{T}} u(\boldsymbol{x}) - P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &+\int_\Omega \nabla(\Delta u - \mathcal{I}_{\mathcal{T}}\Delta u)(\boldsymbol{x})\cdot\nabla w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Using (14) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\int_\Omega (\Delta_{\mathcal{T}} v(\boldsymbol{x}) - \Delta_{\mathcal{T}} u_{\mathcal{T}}(\boldsymbol{x}))\Delta_{\mathcal{T}} w(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\\ &\le \Big(\|\Delta_{\mathcal{T}} v - \overline{\Delta}_{\mathcal{T}} u\|_{L^2(\Omega)} + \|\overline{\Delta}_{\mathcal{T}} u - P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)} + 2\mathrm{diam}(\Omega)\|\nabla(\Delta u - \mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)^d}\Big)\|\Delta_{\mathcal{T}} w\|_{L^2(\Omega)}.\end{aligned}$$

Taking $w = v - u_{\mathcal{T}}$ in the above inequality, we get

$$\begin{aligned}\|\Delta_{\mathcal{T}} v - \Delta_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2(\Omega)} \le\ &\|\Delta_{\mathcal{T}} v - \overline{\Delta}_{\mathcal{T}} u\|_{L^2(\Omega)} + \|\overline{\Delta}_{\mathcal{T}} u - P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)}\\ &+2\ \mathrm{diam}(\Omega)\|\nabla(\Delta u - \mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)^d}.\end{aligned}$$

We now write

$$\|\Delta u - \Delta_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2(\Omega)} \le \|\Delta u - \overline{\Delta}_{\mathcal{T}} u\|_{L^2(\Omega)} + \|\overline{\Delta}_{\mathcal{T}} u - \Delta_{\mathcal{T}} v\|_{L^2(\Omega)} + \|\Delta_{\mathcal{T}} v - \Delta_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2(\Omega)}.$$

Thanks to the two above inequalities, we get

$$\begin{aligned}\|\Delta u - \Delta_{\mathcal{T}} u_{\mathcal{T}}\|_{L^2(\Omega)} \le\ &2\|\overline{\Delta}_{\mathcal{T}} u - \Delta_{\mathcal{T}} v\|_{L^2(\Omega)} + \|\Delta u - \overline{\Delta}_{\mathcal{T}} u\|_{L^2(\Omega)} + \|\overline{\Delta}_{\mathcal{T}} u - P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)}\\ &+2\ \mathrm{diam}(\Omega)\|\nabla(\Delta u - \mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)^d}.\end{aligned}$$

Thanks to the regularity of Δu, we have

$$\begin{aligned}\|\Delta u - \overline{\Delta}_{\mathcal{T}} u\|^2_{L^2(\Omega)} &= \sum_{\boldsymbol{z}\in\mathcal{V}} \int_{K_{\boldsymbol{z}}} \left(\frac{1}{|K_{\boldsymbol{z}}|}\int_{K_{\boldsymbol{z}}} (\Delta u(\boldsymbol{y}) - \Delta u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right)^2 \mathrm{d}\boldsymbol{y}\\ &\le |\Omega|\ 4\ h_{\mathcal{T}}^2 4\|u\|^2_{\infty,3}.\end{aligned} \tag{39}$$

We may also write

$$\begin{aligned}\|\overline{\Delta}_{\mathcal{T}} u - P_{\mathcal{T}}(\mathcal{I}_{\mathcal{T}}\Delta u)\|^2_{L^2(\Omega)} &= \sum_{\boldsymbol{z}\in\mathcal{V}} |K_{\boldsymbol{z}}| \left(\frac{1}{|K_{\boldsymbol{z}}|}\int_{K_{\boldsymbol{z}}} (\Delta u(\boldsymbol{x}) - \Delta u(\boldsymbol{z}))\mathrm{d}\boldsymbol{x}\right)^2\\ &\le |\Omega|\ h_{\mathcal{T}}^2 4\|u\|^2_{\infty,3}.\end{aligned}$$

Applying standard results on the interpolation error of the regular function Δu in $V_{\mathcal{T}}$, we have the existence of C_{P1}, only depending on θ and Ω, such that

$$\|\nabla(\Delta u - \mathcal{I}_{\mathcal{T}}\Delta u)\|_{L^2(\Omega)^d} \le C_{P1} h_{\mathcal{T}}\|u\|_{\infty,4}.$$

Therefore the proof of (38) follows. □
We can then state the following result.

Theorem 4.2 (Error estimate in the case where $u \in C_c^4(\Omega)$)
Let us assume Hypotheses (3) and (4), let $u \in C_c^4(\Omega)$ be given and let $f = \Delta(\Delta u)$. Let $\mathcal{T}$ be a conforming simplicial mesh of Ω and let $\theta > 0$ be such that $\theta < \theta_{\mathcal{T}}$. Let $u_{\mathcal{T}} \in V_{\mathcal{T},0}$ be the solution of (10) in the case where $f = \Delta(\Delta u)$, $\boldsymbol{g} = 0$ and $\ell = 0$. Then there exists $C > 0$, only depending on Ω, θ and u such that

$$\|u_{\mathcal{T}} - u\|_{L^2(\Omega)} \le C h_{\mathcal{T}}, \tag{40}$$

$$\|\nabla u_{\mathcal{T}} - \nabla u\|_{L^2(\Omega)^d} \le C h_{\mathcal{T}}, \tag{41}$$

and

$$\|\Delta_{\mathcal{T}} u_{\mathcal{T}} - \Delta u\|_{L^2(\Omega)} \le C h_{\mathcal{T}}. \tag{42}$$

PROOF. We apply Lemma 4.1 with $v = \widetilde{\mathcal{I}}_{\mathcal{T}} u$. Thanks to Lemma 3.6, we get the existence of $C > 0$, only depending on Ω, θ and u (by its derivatives, and the distance of the support of u to the boundary of the domain) such that (42) holds. Then, writing

$$\|\nabla u_{\mathcal{T}} - \nabla u\|_{L^2(\Omega)^d} \le \|\nabla u_{\mathcal{T}} - \nabla \widetilde{\mathcal{I}}_{\mathcal{T}} u\|_{L^2(\Omega)^d} + \|\nabla \widetilde{\mathcal{I}}_{\mathcal{T}} u - \nabla u\|_{L^2(\Omega)^d},$$

we can apply Lemma 3.2. We get

$$\begin{aligned} \|\nabla u_{\mathcal{T}} - \nabla u\|_{L^2(\Omega)^d} &\le 2 \operatorname{diam}(\Omega) \|\Delta_{\mathcal{T}} u_{\mathcal{T}} - \Delta_{\mathcal{T}} \widetilde{\mathcal{I}}_{\mathcal{T}} u\|_{L^2(\Omega)^d} + \|\nabla \widetilde{\mathcal{I}}_{\mathcal{T}} u - \nabla u\|_{L^2(\Omega)^d} \\ &\le 2 \operatorname{diam}(\Omega) (\|\Delta_{\mathcal{T}} u_{\mathcal{T}} - \Delta u\|_{L^2(\Omega)^d} + \|\Delta u - \Delta_{\mathcal{T}} \widetilde{\mathcal{I}}_{\mathcal{T}} u\|_{L^2(\Omega)^d}) \\ &\quad + \|\nabla \widetilde{\mathcal{I}}_{\mathcal{T}} u - \nabla u\|_{L^2(\Omega)^d}. \end{aligned}$$

Using (42) and Lemma 3.6 provide (41). Then (40) results from the Poincaré inequality. □
Let us now state the result, without assuming that the solution has a compact support.

Theorem 4.3 (Error estimate in the case where $u \in C^4(\overline{\Omega}) \cap H_0^2(\Omega)$)
Let us assume Hypotheses (3) *and* (4)*, let $u \in C^4(\overline{\Omega}) \cap H_0^2(\Omega)$ be given and let $f = \Delta(\Delta u)$. Let $\mathcal{T}$ be a conforming simplicial mesh of Ω and let $\theta > 0$ be such that $\theta < \theta_{\mathcal{T}}$. Let $u_{\mathcal{T}} \in V_{\mathcal{T},0}$ be the solution of* (10) *in the case where $f = \Delta(\Delta u)$, $\boldsymbol{g} = 0$ and $\ell = 0$. Then there exists $C > 0$, only depending on Ω, θ and u such that*

$$\|u_{\mathcal{T}} - u\|_{L^2(\Omega)} \le C h_{\mathcal{T}}^{\frac{1}{5}}, \tag{43}$$

$$\|\nabla u_{\mathcal{T}} - \nabla u\|_{L(\Omega)^d} \le C h_{\mathcal{T}}^{\frac{1}{5}}, \tag{44}$$

and

$$\|\Delta_{\mathcal{T}} u_{\mathcal{T}} - \Delta u\|_{L^2(\Omega)} \le C h_{\mathcal{T}}^{\frac{1}{5}}. \tag{45}$$

PROOF.
For a given $a > 0$, we define the function ψ_a by (29), and the function u_a by

$$u_a(\boldsymbol{x}) = u(\boldsymbol{x}) \psi_a(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Omega.$$

We remark that, for any $i, j = 1, \ldots, d$,

$$\partial_{ij}^2 u_a(\boldsymbol{x}) = \psi_a(\boldsymbol{x}) \partial_{ij}^2 u(\boldsymbol{x}) + \partial_i \psi_a(\boldsymbol{x}) \partial_j u(\boldsymbol{x}) + \partial_j \psi_a(\boldsymbol{x}) \partial_i u(\boldsymbol{x}) + u(\boldsymbol{x}) \partial_{ij}^2 \psi_a(\boldsymbol{x}). \tag{46}$$

Thanks to a Taylor expansion of u and ∇u from any point $\boldsymbol{y} \in \partial\Omega$ such that $|\boldsymbol{x} - \boldsymbol{y}| = \mathrm{d}(\boldsymbol{x}, \partial\Omega)$, we get the existence of $C_u > 0$, only depending on u such that, for all $\boldsymbol{x} \in \Omega$, $|\nabla u(\boldsymbol{x})| \le C_u \mathrm{d}(\boldsymbol{x}, \partial\Omega)$ and $|u(\boldsymbol{x})| \le C_u \mathrm{d}(\boldsymbol{x}, \partial\Omega)^2$. Thanks to $|\partial_i \psi_a(\boldsymbol{x})| \le C/a$ and $|\partial_{ij}^2 \psi_a(\boldsymbol{x})| \le C/a^2$, we get from (46) the existence of C_u', only depending on u, such that

$$|u_a|_2 \le C_u', \ \forall a \in [0, \operatorname{diam}(\Omega)]. \tag{47}$$

Hence we have the existence of C_u'', only depending on u,

$$|\Delta u_a(\boldsymbol{x}) - \Delta u(\boldsymbol{x})| \le C_u'', \ \forall \boldsymbol{x} \in \Omega \text{ such that } \mathrm{d}(x, \partial\Omega) \le a.$$

Since $\Delta u_a(\boldsymbol{x}) = \Delta u(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$ such that $\mathrm{d}(\boldsymbol{x}, \partial\Omega) \ge a$, we get

$$\|\Delta u_a(\boldsymbol{x}) - \Delta u(\boldsymbol{x})\|_{L^2(\Omega)}^2 \le |\{\boldsymbol{x} \in \Omega, \mathrm{d}(\boldsymbol{x}, \partial\Omega) \le a\}| \ (C_u'')^2.$$

Thanks to Hypotheses (3), there exists some $C_\Omega > 0$ such that

$$|\{\boldsymbol{x} \in \Omega, \mathrm{d}(\boldsymbol{x}, \partial\Omega) \le a\}| \le C_\Omega a, \ \forall a \in [0, \operatorname{diam}(\Omega)].$$

On the other hand, since the distance between the support of $u_a \in C_c^\infty(\Omega)$ and $\partial\Omega$ is greater than $a/4$, Lemma 3.6 gives the existence of $C_7 > 0$, only depending on Ω and θ, such that

$$\|\Delta_{\mathcal{T}}\widetilde{\mathcal{I}}_{\mathcal{T}} u_a - \overline{\Delta}_{\mathcal{T}} u_a\|_{L^2(\Omega)} \le C_7 \frac{h_{\mathcal{T}}|u_a|_2}{a^2} \le C_7 \frac{h_{\mathcal{T}} C'_u}{a^2}.$$

We have

$$\|\Delta_{\mathcal{T}}\widetilde{\mathcal{I}}_{\mathcal{T}} u_a - \Delta u\|_{L^2(\Omega)} \le \|\Delta_{\mathcal{T}}\widetilde{\mathcal{I}}_{\mathcal{T}} u_a - \overline{\Delta}_{\mathcal{T}} u_a\|_{L^2(\Omega)} + \|\overline{\Delta}_{\mathcal{T}} u_a - \overline{\Delta}_{\mathcal{T}} u\|_{L^2(\Omega)} + \|\overline{\Delta}_{\mathcal{T}} u - \Delta u\|_{L^2(\Omega)}.$$

Thanks to the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}\|\overline{\Delta}_{\mathcal{T}} u_a - \overline{\Delta}_{\mathcal{T}} u\|_{L^2(\Omega)}^2 &= \sum_{\boldsymbol{z}\in\mathcal{V}} |K_{\boldsymbol{z}}| \left(\frac{1}{|K_{\boldsymbol{z}}|}\int_{K_{\boldsymbol{z}}} (\Delta u_a(\boldsymbol{x}) - \Delta u(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\right)^2 \\ &\le \sum_{\boldsymbol{z}\in\mathcal{V}} \int_{K_{\boldsymbol{z}}} (\Delta u_a(\boldsymbol{x}) - \Delta u(\boldsymbol{x}))^2 \mathrm{d}\boldsymbol{x} = \|\Delta u_a - \Delta u\|_{L^2(\Omega)}^2.\end{aligned}$$

Since (39) provides

$$\|\overline{\Delta}_{\mathcal{T}} u - \Delta u\|_{L^2(\Omega)} \le 4\ |\Omega|^{1/2} h_{\mathcal{T}} \|u\|_{\infty,3},$$

we then get the existence of $C_8 > 0$ independent on the mesh, such that

$$\|\Delta_{\mathcal{T}}\widetilde{\mathcal{I}}_{\mathcal{T}} u_a - \Delta u\|_{L^2(\Omega)} \le C_7 \frac{h_{\mathcal{T}} C'_u}{a^2} + (C_\Omega a)^{1/2}\ (C''_u) + C_8 h_{\mathcal{T}}.$$

Choosing $a_0 = h_{\mathcal{T}}^{2/5}$ leads to the existence of $C_9 > 0$, only depending on u, Ω and θ, such that

$$\|\Delta_{\mathcal{T}}\widetilde{\mathcal{I}}_{\mathcal{T}} u_{a_0} - \Delta u\|_{L^2(\Omega)} \le C_9 h_{\mathcal{T}}^{1/5}.$$

Applying Lemma 4.1 with $v = \widetilde{\mathcal{I}}_{\mathcal{T}} u_{a_0}$, we conclude the proof of the theorem, following the proof of Theorem 4.2 for the derivation of (44) and (43). □

5 Numerical results

Let us introduce the following error norms for the solution, its gradient and its Laplacian:

$$E_0 = \left(\sum_{\boldsymbol{z}\in\mathcal{V}} |K_{\boldsymbol{z}}|(u(\boldsymbol{z}) - u_{\mathcal{T}}(\boldsymbol{z}))^2\right)^{1/2} / \|u\|_{L^2(\Omega)},$$

$$E_1 = \left(\sum_{S\in\mathcal{T}} |S|\,|\nabla_S u_{\mathcal{T}} - \nabla u(\boldsymbol{x}_S)|^2\right)^{1/2} / \|\nabla u\|_{L^2(\Omega)},$$

denoting by $\boldsymbol{x}_S$ the center of gravity of S, and

$$E_2 = \left(\sum_{\boldsymbol{z}\in\mathcal{V}} |K_{\boldsymbol{z}}|(\Delta_{\boldsymbol{z}} u_{\mathcal{T}} - \Delta u(\boldsymbol{z}))^2\right)^{1/2} / \|\Delta u\|_{L^2(\Omega)}.$$

We are going to study these error norms for a one-dimensional and a two-dimensional examples.

5.1 One-dimensional case

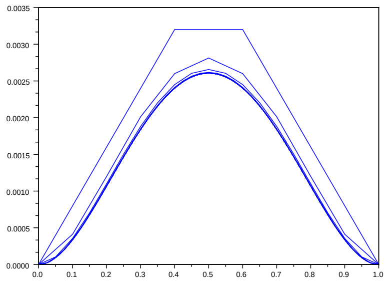
We approximate the solution

$$u(x) = \frac{(x(1-x))^2}{24}, \ \forall x \in [0,1],$$

of the problem

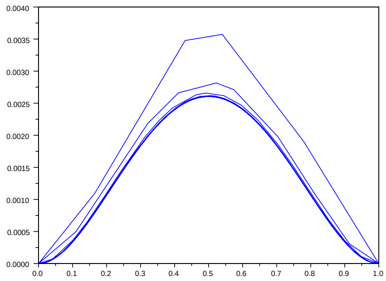
$$u^{(4)}(x) = 1, \ x \in [0,1],$$
$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

using Scheme (10), with different 1D meshes with N interior points. In the following figure and table, we show the numerical results obtained in the case where the mesh is uniform, i.e. the points $\boldsymbol{z} \in \mathcal{V}$ are located at the abscissae i/N, for $i = 0, \ldots, N$.



N	E_0	order	E_1	order	E_2	order
5	0.366	-	0.246	-	8.94E-2	-
10	9.16E-2	$\simeq$2	6.24E-2	$\simeq$2	2.24E-2	$\simeq$2
20	2.29E-2	$\simeq$2	1.57E-2	$\simeq$2	5.59E-3	$\simeq$2
40	5.73E-3	$\simeq$2	3.92E-3	$\simeq$2	1.40E-3	$\simeq$2
80	1.43E-3	$\simeq$2	9.80E-4	$\simeq$2	3.49E-4	$\simeq$2
160	3.58E-4	$\simeq$2	2.45E-4	$\simeq$2	8.73E-5	$\simeq$2
320	8.95E-5	$\simeq$2	6.13E-5	$\simeq$2	2.18E-5	$\simeq$2
640	2.25E-5	$\simeq$2	1.54E-5	$\simeq$2	5.50E-6	$\simeq$2

An order 2 is numerically obtained for the solution, its gradient and its discrete Laplacian, which is much more than the theoretical order proved in this case (1/5). In the following figure and table, we show the numerical results obtained when the interior points $\boldsymbol{z} \in \mathcal{V}$ are located at the abscissae $(i + \alpha_i)/N$, for $i = 1, \ldots, N-1$, where α_i is a random value between -0.3 and 0.3.



N	E_0	order	E_1	order	E_2	order
5	0.416	-	0.283	-	0.111	-
10	9.66E-2	$\simeq$2	6.64E-2	$\simeq$2	2.49E-2	$\simeq$2
20	2.40E-2	$\simeq$2	1.70E-2	$\simeq$2	6.24E-3	$\simeq$2
40	6.45E-3	$\simeq$2	4.67E-3	$\simeq$2	1.62E-3	$\simeq$2
80	1.64E-3	$\simeq$2	1.19E-3	$\simeq$2	4.15E-4	$\simeq$2
160	4.42E-4	$\simeq$2	3.31E-4	$\simeq$2	1.05E-5	$\simeq$2
320	1.03E-5	$\simeq$2	7.56E-5	$\simeq$2	2.53E-5	$\simeq$2
640	2.70E-6	$\simeq$2	2.02E-5	$\simeq$2	6.50E-6	$\simeq$2

Again, an order 2 is numerically obtained for the solution, its gradient and its discrete Laplacian, which shows the robustness of the scheme in this less regular case.

5.2 Two-dimensional cases

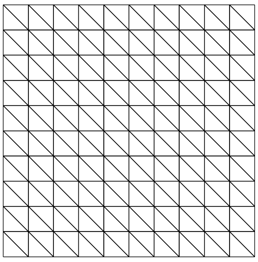
We consider Scheme (10) for the approximation of the 2D problem, where $\Omega = (0,1)^2$ and where the continuous solution is given by:

$$u(x_1, x_2) = (1 - \cos(2\pi x_1))(1 - \cos(2\pi x_2)), \ \forall (x_1, x_2) \in [0,1]^2,$$

which satisfies (2) for the *ad hoc* data $f = \Delta(\Delta u)$, $\boldsymbol{g} = 0$, $\ell = 0$, $\Omega =]0,1[^2$; hence we choose f as the function defined by:

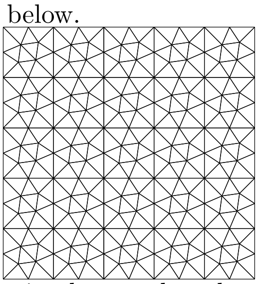
$$f(x_1, x_2) = \Delta(\Delta u)(x_1, x_2) = (2\pi)^4 \big(4\cos(2\pi x_1)\cos(2\pi x_2) - (\cos(2\pi x_1) + \cos(2\pi x_2))\big)$$

We first consider the case where the mesh is obtained by splitting $N_{\mathrm{raf}} \times N_{\mathrm{raf}}$ squares in 2 triangles. The total number of triangles is then 2 N_{raf}^2, and the size of the mesh is of order $1/N_{\mathrm{raf}}$. In the figure below, we show one of the meshes used, and in the table below, we present the results obtained using the scheme (10).



N_{raf}	E_0	order	E_1	order	E_2	order	$u_{\min}$	$u_{\max}$
10	6.82E-2	-	0.171	-	3.36E-2	-	0.000	4.273
20	1.66E-2	$\simeq$2	8.14E-2	$\simeq$1	8.27E-3	$\simeq$2	0.000	4.066
40	4.12E-3	$\simeq$2	4.02E-2	$\simeq$1	2.06E-3	$\simeq$2	0.000	4.016
80	1.03E-3	$\simeq$2	2.00E-2	$\simeq$1	5.14E-4	$\simeq$2	0.000	4.004
160	2.57E-4	$\simeq$2	1.00E-2	$\simeq$1	1.29E-4	$\simeq$2	0.000	4.001

In this 2D case, we again observe that the order of convergence is much better than 1/5. Turning to less regular meshes, we consider the case where the simplicial meshes are generated by the repetition of the same square pattern. In the figure below, one can see the repetition of $N_{\mathrm{raf}} \times N_{\mathrm{raf}}$ times the initial pattern, with $N_{\mathrm{raf}} = 5$. The total number of triangles is then 14 N_{raf}^2, and the size of the mesh is of order $1/N_{\mathrm{raf}}$. The interest of such meshes is that no symmetry can increase the numerical order of convergence, whereas the regularity factor of the mesh remains constant. We then observe the results provided in the table below.



N_{raf}	E_0	order	E_1	order	E_2	order	$u_{\min}$	$u_{\max}$
5	5.06E-2	-	0.101	-	5.50E-2	-	0.000	4.125
10	1.16E-2	> 2	4.60E-2	$\simeq 1$	3.29E-2	< 1	0.000	4.059
20	2.49E-3	> 2	2.25E-2	$\simeq 1$	2.27E-2	< 1	0.000	4.013
40	4.41E-4	> 2	1.12E-2	$\simeq 1$	1.60E-2	< 1	0.000	4.003
80	8.27E-5	> 2	5.57E-3	$\simeq 1$	1.13E-2	< 1	0.000	4.000

We again observe that the numerical orders of convergence are much better than the theoretical ones for E_0 and E_1, but only slightly better for E_2.

6 Conclusion

We show in this paper that it is possible to approximate the solution in $H_0^2(\Omega)$ of the biharmonic problem using a P1 finite element approximation, which results in a robust and cheap scheme. Since the approximate solution only belongs to $H_0^1(\Omega)$, a discrete Laplace operator is used in the discrete variational formulation. This operator, applied to the natural interpolation of a regular function, is not consistent with the continuous Laplace operator, and an adapted interpolation is provided. This allows to prove the convergence of the scheme in the general case, and to derive error estimates. Numerical observations show that these error estimates are not sharp.
Hence some further work should explore possible improvements of these error estimates. The problem of the approximation of more general fourth-order elliptic operators by P1 finite elements should also be examined.

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